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Asymptotics of the Farey Fraction Spin Chain Free Energy at the Critical Point

Oscar F. Bandtlow · Jan Fiala · Peter Kleban · Thomas Prellberg

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Abstract We consider the Farey fraction spin chain in an external field h. Using ideas from dynamical systems and functional analysis, we show that the free energy f in the vicinity of the second-order phase transition is given, exactly, by

$$f \sim \frac{t}{\log t} - \frac{1}{2}\frac{h^2}{t} \quad \text{for } h^2 \ll t \ll 1.$$

Here $t = \lambda_G \log(2)(1 - \frac{\beta}{\beta_c})$ is a reduced temperature, so that the deviation from the critical point is scaled by the Lyapunov exponent of the Gauss map, λ_G. It follows that λ_G determines the amplitude of both the specific heat and susceptibility singularities. To our knowledge, there is only one other microscopically defined interacting model for which the free energy near a phase transition is known as a function of two variables.

Our results confirm what was found previously with a cluster approximation, and show that a clustering mechanism is in fact responsible for the transition. However, the results disagree in part with a renormalisation group treatment.

Keywords Phase transition · Farey fractions · Spin chain · Transfer operator

O.F. Bandtlow · T. Prellberg (✉)
School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, UK
e-mail: t.prellberg@qmul.ac.uk

O.F. Bandtlow
e-mail: o.bandtlow@qmul.ac.uk

J. Fiala
Department of Chemistry and Physics, Lamar University, P.O. Box 10046, Beaumont, TX 77710, USA
e-mail: jfiala@my.lamar.edu

P. Kleban
LASST and Department of Physics & Astronomy, University of Maine, Orono, ME 04469, USA
e-mail: kleban@maine.edu

1 Introduction

The theory of second-order phase transitions has a long and well-developed history. However, for spin models coupled to an external magnetic field there are very few exact microscopic calculations for the free energy $f(\beta, h)$ as a function of both inverse temperature β *and* magnetic field strength h in the vicinity of such a transition, and (to our knowledge) with the exception of the ice-rule models (see Sect. V.E. of [1]), these are limited to mean-field, non-interacting models (e.g. the spherical and Kac-van der Waals models [2]), or other similarly defined models [3–5].

In this work, by using operator techniques, we calculate the exact free energy $f(\beta, h)$ for a model with many-body long-range interactions [6]. This model has been investigated previously with a cluster approximation [7], replacing the non-trivial many-body interactions by interactions within clusters, and thereby leading to a model very similar to [3–5]. Our present work confirms the validity of that cluster approximation near the critical point. Intriguingly, we also find that both critical amplitudes in the free energy scale with a Lyapunov exponent.

Phase transitions in one-dimensional systems are unusual, essentially because, as long as the interactions are of finite range and strength, any putative ordered state at finite temperature will be disrupted by thermally induced defects, and a defect in one dimension is very effective at destroying order. On the other hand, long range or infinite interactions generally make the model ordered at all finite temperatures. Despite this, there are a number of examples of one-dimensional systems that exhibit a phase transition. The Farey fraction spin chain [6] is one such case, which has attracted interest from both physicists and mathematicians (see [8–12] and references therein). This model has a phase transition at a finite temperature. While the transition is of second-order, it has some properties that are usually found with a first-order transition: for external field $h = 0$, the magnetisation jumps from completely saturated, below the transition, to zero above it. Despite this unusual behaviour, the model does not violate scaling theory, but rather is encompassed as a limiting case [13].

In some recent work, [7, 13], this model has been generalised to finite external field, and analysed via both renormalisation group methods and with a dynamical system-inspired cluster approximation. Neither method is rigorous, and the results are not quite the same. Specifically, the dependence of $f(\beta, h)$ on h differs by logarithmic factors. This motivates a more rigorous analysis of the model. We find that the cluster picture indeed leads to the correct result for the asymptotic form for $f(\beta, h)$, and, in addition, we are able to evaluate the constants. Intriguingly, they involve λ_G, the Lyapunov exponent of the Gauss map. This arises naturally here, since the Gauss map is intimately related to the first-return map of the Farey map, which specifies the transfer operator giving the Farey fraction spin chain partition function [8].

In Sect. 2 we define the model, first in the standard way using matrices, and then via transfer operators. In Sect. 3 we derive some operator identities that are necessary for our analysis, and in Sect. 4 we specify a function space and study the spectral properties of the relevant transfer operators. Section 5 describes the connection to the Gauss map. Section 6 is the heart of our work. Here we use perturbation theory around the critical point $(\beta, h) = (1, 0)$ to find the asymptotic behaviour of the free energy $f(\beta, h)$. The key to our method is the use of a "cluster operator", which encodes the behaviour of clusters of up and down spins while possessing tractable spectral properties, thus validating the results obtained with the cluster approximation of [7].

2 The Model

The Farey fraction spin chain may be constructed, for inverse temperature β and magnetic field h, via the two matrices

$$A_\uparrow = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A_\downarrow = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{1}$$

which correspond to "spin up" and "spin down", respectively. The spin chain partition function comes in various guises, all of which have the same free energy (at least for $h = 0$, see [6, 8, 13, 14]). Here we are considering the generalised Knauf spin chain [8], not the "trace" model studied in [7]. We make this choice for technical reasons. However, by universality, our results are supposed to apply to any of the Farey spin chains (see [6, 8, 13, 14] for definitions of the various chains).

Defining matrix products

$$M_N := \prod_{i=1}^{N} A_\uparrow^{1-\sigma_i} A_\downarrow^{\sigma_i}, \quad \sigma_i \in \{0, 1\}, \tag{2}$$

where the dependence of M_N on the σ_i has been suppressed, and writing a given matrix product as $M_N = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we define the spin chain partition function by

$$Z_N(\beta, h; x) = \sum_{\{\sigma_i\}} \frac{1}{(cx + d)^{2\beta}} e^{-\beta h(2\sum_{i=1}^{N} \sigma_i - N)}, \tag{3}$$

where $x \geq 0$ is a parameter (which does not affect the free energy, as we shall argue below). When M_N begins with $A_\uparrow$, c and d are neighbouring Farey denominators at level N in the modified Farey sequence (see [6] for further details on this connection), whence the nomenclature "Farey" for this spin chain model.

The free energy $f(\beta, h)$ is defined via

$$-\beta f(\beta, h) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta, h; x). \tag{4}$$

Alternatively, the partition function can be expressed using transfer operators. In order to emphasise the difference between matrices and operators, we denote the latter with script letters. We begin by defining the operator

$$\mathcal{L}_{\beta,h} = e^{-\beta h} \mathcal{L}_\beta^\uparrow + e^{\beta h} \mathcal{L}_\beta^\downarrow \tag{5}$$

where

$$\mathcal{L}_\beta^\uparrow f(x) = \frac{1}{(1 + x)^{2\beta}} f\left(\frac{x}{1 + x}\right) \quad \text{and} \quad \mathcal{L}_\beta^\downarrow f(x) = f(1 + x). \tag{6}$$

Thus we obtain, as in [8]

$$Z_N(\beta, h; x) = \mathcal{L}_{\beta,h}^N 1(x). \tag{7}$$

This expression indicates that the free energy $f(\beta, h)$ is given by the logarithm of the spectral radius of $\mathcal{L}_{\beta,h}$ on a suitable function space,

$$-\beta f(\beta, h; x) = \log r(\mathcal{L}_{\beta,h}). \tag{8}$$

We will return to this point below when we specify the function space used in our analysis.

There is another notation that is sometimes used in the literature, which we mention for completeness and for comparison with previous work on this model. Using a "slash" notation which is standard in number theory, the action of a 2×2 matrix $[M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})]$ on a function f is defined via

$$f(x)\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right. = \frac{1}{(cx+d)^{2\beta}} f\left(\frac{ax+b}{cx+d}\right). \tag{9}$$

Thus (3) can be written as

$$Z_N(\beta, h; x) = 1(x)|(e^{-\beta h} A_\uparrow + e^{\beta h} A_\downarrow)^N. \tag{10}$$

Note that in order to be consistent with the group structure of $SL_2(\mathbb{Z})$, any addition and scalar multiplication is performed *after* the matrix action on the function has been computed.

In the disordered (high-temperature) phase, we expect that there is a leading eigenvalue $\lambda(\beta, h)$ of $\mathcal{L}_{\beta,h}$ which satisfies $\lambda(\beta, h) > 1$, is non-degenerate, and belongs to the discrete spectrum, and that the free energy is given by

$$-\beta f(\beta, h; x) = \log \lambda(\beta, h), \tag{11}$$

which is independent of x.

There is an obvious (spin flip) symmetry in our model. Since

$$S A_\uparrow S = A_\downarrow \quad \text{with } S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{12}$$

it is natural to define the corresponding operator $\mathcal{S}_\beta$ as

$$\mathcal{S}_\beta f(x) = x^{-2\beta} f(1/x). \tag{13}$$

Note that $S^{-1} = S$ and $\mathcal{S}_\beta^{-1} = \mathcal{S}_\beta$, i.e. both S and $\mathcal{S}_\beta$ are involutions. For the transfer operators we find

$$\mathcal{L}_\beta^\uparrow = \mathcal{S}_\beta \mathcal{L}_\beta^\downarrow \mathcal{S}_\beta \quad \text{and} \quad \mathcal{L}_{\beta,h} = \mathcal{S}_\beta \mathcal{L}_{\beta,-h} \mathcal{S}_\beta. \tag{14}$$

The operator $\mathcal{L}_{\beta,0}$ has a nice interpretation as the Ruelle-Perron-Frobenius transfer operator [15] for the dynamical system given by iteration of the map

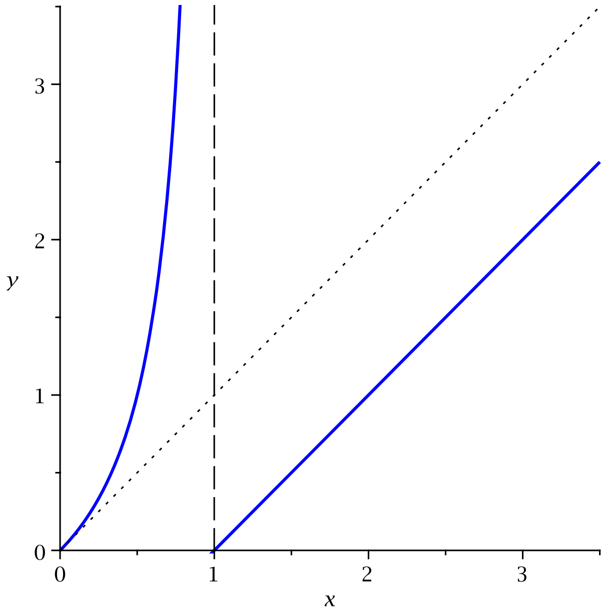
$$T(x) = \begin{cases} x/(1-x), & 0 \le x < 1 \\ x - 1, & x \ge 1, \end{cases} \tag{15}$$

and $\mathcal{L}_{\beta,h}$ can be viewed as a weighted generalisation. Note that the map T has the symmetry $T(1/x) = 1/T(x)$. This map differs from the Farey map used in, say, [8]. The graph of this map is shown in Fig. 1. One clearly observes the existence of two marginal neutral fixed points, at the origin and infinity.

Now consider the generating function

$$G(\beta, h, z; x) = \sum_{N=0}^{\infty} z^N Z_N(\beta, h; x). \tag{16}$$

Fig. 1 The graph of the map $y = T(x)$



Examination of G motivates the operator relations discussed below and makes a connection with the treatment in [7]. One may rewrite G in terms of the resolvent $[1 - z\mathcal{L}_{\beta,h}]^{-1}$ of $\mathcal{L}_{\beta,h}$ as

$$G(\beta, h, z; x) = [1 - z\mathcal{L}_{\beta,h}]^{-1}1(x). \tag{17}$$

Equation (17) indicates that $z_c(\beta, h)$ is equal to the inverse of the spectral radius $1/r(\mathcal{L}_{\beta,h})$. The free energy is then given as

$$\beta f(\beta, h) = \log z_c(\beta, h), \tag{18}$$

where $z_c(\beta, h)$ is the smallest singularity of $G(\beta, h, z; x)$ in z on the positive real axis. Thus, in principle, we could find the free energy by analysing G. However, it is very difficult to do this directly, since $\mathcal{L}_{\beta,h}$ is not sufficiently well-behaved. In order to make progress we resort below to a more nuanced treatment.

3 Identities and Spectral Relations

This section introduces a Lemma that is the basis of our analysis. It allows us to avoid dealing directly with $\mathcal{L}_{\beta,h}$, which is difficult to control at the critical point $(\beta, h) = (1, 0)$.

To motivate this section, let us consider an arrangement of N spins. We can collect adjacent spins of equal orientation into clusters of the form

$$\underbrace{\uparrow\cdots\uparrow}_{\geq 0}\overbrace{\underbrace{\downarrow\cdots\downarrow}_{\geq 1}\uparrow\cdots\downarrow\underbrace{\uparrow\cdots\uparrow}_{\geq 1}}^{n \text{ pairs},\, n\geq 0}\underbrace{\downarrow\cdots\downarrow}_{\geq 0}.$$

Notice that configurations starting and ending with either spin orientation are included. Such an arrangement of spins corresponds uniquely to a particular product of operators $\mathcal{L}_\beta^\uparrow$ and $\mathcal{L}_\beta^\downarrow$,

$$\underbrace{\mathcal{L}_\beta^\uparrow \cdots \mathcal{L}_\beta^\uparrow}_{\geq 0} \overbrace{\underbrace{\mathcal{L}_\beta^\downarrow \cdots \mathcal{L}_\beta^\downarrow}_{\geq 1} \mathcal{L}_\beta^\uparrow \cdots \mathcal{L}_\beta^\downarrow \underbrace{\mathcal{L}_\beta^\uparrow \cdots \mathcal{L}_\beta^\uparrow}_{\geq 1}}^{n \text{ pairs}, \, n \geq 0} \underbrace{\mathcal{L}_\beta^\downarrow \cdots \mathcal{L}_\beta^\downarrow}_{\geq 0},$$

and taking a weighted sum over all possible spin configurations of arbitrary length, we find

$$\begin{aligned}
[1 - z\mathcal{L}_{\beta,h}]^{-1} &= [1 - ze^{-\beta h}\mathcal{L}_\beta^\uparrow - ze^{\beta h}\mathcal{L}_\beta^\downarrow]^{-1} \\
&= [1 - ze^{-\beta h}\mathcal{L}_\beta^\uparrow]^{-1} \sum_{n=0}^{\infty} \big(ze^{\beta h}\mathcal{L}_\beta^\downarrow [1 - ze^{\beta h}\mathcal{L}_\beta^\downarrow]^{-1} ze^{-\beta h}\mathcal{L}_\beta^\uparrow [1 - ze^{-\beta h}\mathcal{L}_\beta^\uparrow]^{-1}\big)^n \\
&\quad \times [1 - ze^{\beta h}\mathcal{L}_\beta^\downarrow]^{-1}.
\end{aligned} \tag{19}$$

We now introduce the operators

$$\mathcal{M}_{\beta,\tau}^\uparrow = \tau\mathcal{L}_\beta^\uparrow [1 - \tau\mathcal{L}_\beta^\uparrow]^{-1} \quad \text{and} \quad \mathcal{M}_{\beta,\tau}^\downarrow = \tau\mathcal{L}_\beta^\downarrow [1 - \tau\mathcal{L}_\beta^\downarrow]^{-1}. \tag{20}$$

Notice that as a formal power series in τ,

$$\mathcal{M}_{\beta,\tau}^\uparrow = \sum_{n=1}^{\infty} \tau^n \mathcal{L}_\beta^{\uparrow n} \quad \text{and} \quad \mathcal{M}_{\beta,\tau}^\downarrow = \sum_{n=1}^{\infty} \tau^n \mathcal{L}_\beta^{\downarrow n}. \tag{21}$$

To keep the discussion general, we shall assume in the following that $\mathcal{L}_\beta^\uparrow$ and $\mathcal{L}_\beta^\downarrow$ are bounded operators on a Banach space, which will be specified later.

Then $\mathcal{M}_{\beta,\tau}^\uparrow$ and $\mathcal{M}_{\beta,\tau}^\downarrow$ exist whenever the resolvents $[1 - \tau\mathcal{L}_\beta^\uparrow]^{-1}$ and $[1 - \tau\mathcal{L}_\beta^\downarrow]^{-1}$ exist, i.e. for $\tau^{-1} \notin \sigma(\mathcal{L}_\beta^\uparrow)$ or $\tau \notin \sigma(\mathcal{L}_\beta^\downarrow)$, respectively (here $\sigma(\mathcal{A})$ denotes the spectrum of the operator $\mathcal{A}$). This motivates the following identities.

Lemma 1 *Let* $z^{-1} \notin (\sigma(e^{-\beta h}\mathcal{L}_\beta^\uparrow) \cup \sigma(e^{\beta h}\mathcal{L}_\beta^\downarrow))$. *Then*

$$[1 + \mathcal{M}_{\beta,ze^{\beta h}}^\downarrow][1 - z\mathcal{L}_{\beta,h}][1 + \mathcal{M}_{\beta,ze^{-\beta h}}^\uparrow] = [1 - \mathcal{M}_{\beta,ze^{\beta h}}^\downarrow \mathcal{M}_{\beta,ze^{-\beta h}}^\uparrow] \tag{22}$$

and

$$[1 - z\mathcal{L}_{\beta,h}] = [1 - z\mathcal{L}_\beta^\downarrow][1 - \mathcal{M}_{\beta,ze^{\beta h}}^\downarrow \mathcal{M}_{\beta,ze^{-\beta h}}^\uparrow][1 - z\mathcal{L}_\beta^\uparrow]. \tag{23}$$

Proof Noting that $[1 + \mathcal{M}_{\beta,ze^{\beta h}}^\downarrow] = [1 - z\mathcal{L}_\beta^\downarrow]^{-1}$ and $[1 + \mathcal{M}_{\beta,ze^{-\beta h}}^\uparrow] = [1 - z\mathcal{L}_\beta^\uparrow]^{-1}$, we calculate directly

$$[1 + \mathcal{M}_{\beta,ze^{\beta h}}^\downarrow][1 - z\mathcal{L}_{\beta,h}][1 + \mathcal{M}_{\beta,ze^{-\beta h}}^\uparrow] - 1 \tag{24}$$

$$= [1 - z\mathcal{L}_\beta^\downarrow]^{-1}[1 - z\mathcal{L}_{\beta,h}][1 - z\mathcal{L}_\beta^\uparrow]^{-1} - 1 \tag{25}$$

$$= [1 - z\mathcal{L}_\beta^\downarrow]^{-1}(1 - z\mathcal{L}_{\beta,h} - [1 - z\mathcal{L}_\beta^\downarrow][1 - z\mathcal{L}_\beta^\uparrow])[1 - z\mathcal{L}_\beta^\uparrow]^{-1} \tag{26}$$

$$= [1 - z\mathcal{L}_\beta^\downarrow]^{-1}[-z\mathcal{L}_\beta^\downarrow z\mathcal{L}_\beta^\uparrow][1 - z\mathcal{L}_\beta^\uparrow]^{-1} \tag{27}$$

$$= -\mathcal{M}_{\beta,ze^{\beta h}}^\downarrow \mathcal{M}_{\beta,ze^{-\beta h}}^\uparrow. \tag{28}$$

Multiplying (22) by $[1 - z\mathcal{L}_\beta^\downarrow]$ and $[1 - z\mathcal{L}_\beta^\uparrow]$ from the left and right, respectively, (23) follows. □

As above, we have the symmetry

$$\mathcal{M}_{\beta,\tau}^\uparrow = \mathcal{S}_\beta \mathcal{M}_{\beta,\tau}^\downarrow \mathcal{S}_\beta. \tag{29}$$

It is helpful to take advantage of this symmetry by defining $\mathcal{M}_{\beta,\tau} = \mathcal{M}_{\beta,\tau}^\downarrow \mathcal{S}_\beta$ so that

$$\mathcal{M}_{\beta,ze^{\beta h}}^\downarrow \mathcal{M}_{\beta,ze^{-\beta h}}^\uparrow = \mathcal{M}_{\beta,ze^{\beta h}} \mathcal{M}_{\beta,ze^{-\beta h}}. \tag{30}$$

Note that the rhs is a square when $h = 0$. Utilising (30) and Lemma 1, we arrive at the following characterisation of eigenvalues and eigenfunctions of $\mathcal{L}_{\beta,h}$.

Proposition 2 *Let $z^{-1} \notin (\sigma(e^{-\beta h}\mathcal{L}_\beta^\uparrow) \cup \sigma(e^{\beta h}\mathcal{L}_\beta^\downarrow))$. If f is an eigenfunction of $\mathcal{M}_{\beta,ze^{\beta h}} \mathcal{M}_{\beta,ze^{-\beta h}}$ with eigenvalue 1, then $[1 + \mathcal{M}_{\beta,ze^{-\beta h}}^\uparrow]f$ is an eigenfunction of $\mathcal{L}_{\beta,h}$ with eigenvalue $\lambda = 1/z$. Conversely, if g is an eigenfunction of $\mathcal{L}_{\beta,h}$ with eigenvalue $\lambda = 1/z$, then $[1 - ze^{\beta h}\mathcal{L}_\beta^\downarrow]g$ is an eigenfunction of $\mathcal{M}_{\beta,ze^{\beta h}} \mathcal{M}_{\beta,ze^{-\beta h}}$ with eigenvalue 1.*

Proof If f is an eigenfunction of $\mathcal{M}_{\beta,ze^{\beta h}} \mathcal{M}_{\beta,ze^{-\beta h}}$ with eigenvalue 1, then by (30) the rhs of (22) acting on f is $[1 - \mathcal{M}_{\beta,ze^{\beta h}}^\downarrow \mathcal{M}_{\beta,ze^{-\beta h}}^\uparrow]f = 0$. Due to the assumption on z, the kernels of both $[1 + \mathcal{M}_{\beta,ze^{\beta h}}^\downarrow] = [1 - ze^{\beta h}\mathcal{L}_\beta^\downarrow]^{-1}$ and $[1 + \mathcal{M}_{\beta,ze^{-\beta h}}^\uparrow] = [1 - ze^{-\beta h}\mathcal{L}_\beta^\uparrow]^{-1}$ are zero, so it follows from (22) that $[1 - z\mathcal{L}_{\beta,h}]g = 0$ with $g = [1 + \mathcal{M}_{\beta,ze^{-\beta h}}^\uparrow]f \neq 0$. The second assertion follows similarly using (23). □

Proposition 2 motivates the definition of the set

$$\Omega_{\beta,h} = \left\{1/z : z \in \mathbb{C} \setminus \left(\{0\} \cup \sigma(e^{-\beta h}\mathcal{L}_\beta^\uparrow) \cup \sigma(e^{\beta h}\mathcal{L}_\beta^\downarrow)\right) \text{ and } 1 \in \sigma(\mathcal{M}_{\beta,ze^{\beta h}} \mathcal{M}_{\beta,ze^{-\beta h}})\right\}. \tag{31}$$

The next proposition relates $\Omega_{\beta,h}$ and $\sigma(\mathcal{L}_{\beta,h})$.

Proposition 3

$$\Omega_{\beta,h} = \sigma(\mathcal{L}_{\beta,h}) \setminus \left(\{0\} \cup \sigma(e^{-\beta h}\mathcal{L}_\beta^\uparrow) \cup \sigma(e^{\beta h}\mathcal{L}_\beta^\downarrow)\right). \tag{32}$$

Proof If $z^{-1} \notin (\sigma(e^{-\beta h}\mathcal{L}_\beta^\uparrow) \cup \sigma(e^{\beta h}\mathcal{L}_\beta^\downarrow))$, then (23) implies that $[1 - z\mathcal{L}_{\beta,h}]^{-1}$ exists if and only if $[1 - \mathcal{M}_{\beta,ze^{\beta h}}^\downarrow \mathcal{M}_{\beta,ze^{-\beta h}}^\uparrow]^{-1}$ exists. But this is equivalent to the definition of $\Omega_{\beta,h}$, as $\Omega_{\beta,h} \cap (\sigma(e^{-\beta h}\mathcal{L}_\beta^\uparrow) \cup \sigma(e^{\beta h}\mathcal{L}_\beta^\downarrow)) = \emptyset$. □

Proposition 3 will allow us to study the spectral properties of $\mathcal{L}_{\beta,h}$ by analysing the spectral properties of $\mathcal{M}_{\beta,ze^{\beta h}} \mathcal{M}_{\beta,ze^{-\beta h}}$.

4 The Function Space

We now specify the function space on which $\mathcal{L}_{\beta,h}$ acts and describe some of its spectral properties on this space.

Let $\Pi = \{z \in \mathbb{C} : \Re z > 0\}$ denote the open right half plane and let $H^\infty(\Pi)$ denote the space of bounded holomorphic functions on Π. Equipped with the norm $\|f\| = \sup_{z\in\Pi} |f(z)|$ the space $H^\infty(\Pi)$ becomes a Banach space.

Observe that if ϕ is a holomorphic self-map of Π and $w \in H^\infty(\Pi)$, then the operator

$$\mathcal{C}_{w,\phi} : H^\infty(\Pi) \to H^\infty(\Pi), \tag{33}$$

$$\mathcal{C}_{w,\phi} f(z) = w(z) f(\phi(z)), \tag{34}$$

known as a *weighted composition operator* (see, for example, [16]), is bounded with operator norm $\|\mathcal{C}_{w,\phi}\| = \|w\|$. To see this, note that if $f \in H^\infty(\Pi)$, then $w \cdot f \circ \phi$ is holomorphic and bounded on Π and

$$\|\mathcal{C}_{w,\phi}\| = \sup_{z\in\Pi} |w(z) f(\phi(z))| \leq \|w\| \cdot \|f\|. \tag{35}$$

Thus $\|\mathcal{C}_{w,\phi}\| \leq \|w\|$. But $\|\mathcal{C}_{w,\phi} 1\| = \|w\|$, so $\|\mathcal{C}_{w,\phi}\| = \|w\|$ as claimed.

Before studying the spectral properties of our operators on $H^\infty(\Pi)$, we require some more notation. We write

$$w_\beta^\uparrow(z) = \frac{1}{(1+z)^{2\beta}}, \qquad \phi^\uparrow(z) = \frac{z}{1+z}, \tag{36}$$

$$w_\beta^\downarrow(z) = 1, \qquad \phi^\downarrow(z) = (1+z), \tag{37}$$

so that

$$\mathcal{L}_\beta^\uparrow = \mathcal{C}_{w_\beta^\uparrow, \phi^\uparrow} \quad \text{and} \quad \mathcal{L}_\beta^\downarrow = \mathcal{C}_{w_\beta^\downarrow, \phi^\downarrow}. \tag{38}$$

We shall now consider the spectral properties of our operators in more detail.

Proposition 4

(i) $\mathcal{L}_\beta^\downarrow$ *is a bounded operator on* $H^\infty(\Pi)$. *Its spectrum is the interval* $[0, 1]$ *with every spectral point being an eigenvalue.*
(ii) $\mathcal{L}_\beta^\uparrow$ *is a bounded operator on* $H^\infty(\Pi)$ *provided that* $\Re\beta \geq 0$. *If* $\beta \geq 0$ *then* $\|\mathcal{L}_\beta^{\uparrow\, n}\| = 1$ *for any* $n \in \mathbb{N}$.

Proof For the proof of (i) observe that $\phi^\downarrow$ is a holomorphic self-map of Π and $w_\beta^\downarrow \in H^\infty(\Pi)$ so $\mathcal{L}_\beta^\downarrow$ is bounded by the argument outlined above. The remaining assertions are proved in [17].

We now turn to the proof of (ii). Again, since $\phi^\uparrow$ is a holomorphic self-map of Π and $w_\beta^\uparrow \in H^\infty(\Pi)$ for $\Re\beta \geq 0$ the operator $\mathcal{L}_\beta^\uparrow$ is bounded by the argument outlined above. Suppose now that $\beta \geq 0$. For the norm calculation of $\mathcal{L}_\beta^\uparrow$ observe that $\|w_\beta^\uparrow\| = 1$. Thus $\|\mathcal{L}_\beta^\uparrow\| = 1$ and it follows that $\|\mathcal{L}_\beta^{\uparrow\, n}\| \leq 1$ for $n \in \mathbb{N}$. It remains to show that $\|\mathcal{L}_\beta^{\uparrow\, n}\| \geq 1$. In order to see this, note that if $f \in H^\infty(\Pi)$ is holomorphic at 0 with $f(0) = 1$, then $\mathcal{L}_\beta^\uparrow f$ is also holomorphic at 0 with $\mathcal{L}_\beta^\uparrow f(0) = 1$. Thus $\mathcal{L}_\beta^{\uparrow\, n} 1(0) = 1$ for any $n \in \mathbb{N}$, and so $\|\mathcal{L}_\beta^{\uparrow\, n}\| \geq 1$ as claimed. □

An immediate consequence of the above is the following.

Corollary 5 *If $\beta \geq 0$ the spectral radii of $\mathcal{L}_\beta^\downarrow$ and $\mathcal{L}_\beta^\uparrow$ are given by*

$$r(\mathcal{L}_\beta^\downarrow) = r(\mathcal{L}_\beta^\uparrow) = 1. \tag{39}$$

The following result will play a crucial role in the study of the spectral properties of $\mathcal{L}_{\beta,h}$.

Proposition 6 *If $\Re\beta \geq 0$ then $\mathcal{L}_\beta^\downarrow \mathcal{L}_\beta^\uparrow$ is compact.*

Proof It is not difficult to see that $\phi^\uparrow \circ \phi^\downarrow(\Pi) = \{z \in \mathbb{C} : |z - \frac{3}{4}| < \frac{1}{4}\}$. Thus $\phi^\uparrow \circ \phi^\downarrow$ maps all of Π into a compact subset of Π, and Montel's Theorem [18, Chap. 1, Proposition 6] implies that $\mathcal{C}_{1,\phi^\uparrow\circ\phi^\downarrow}$ is a compact operator on $H^\infty(\Pi)$. But

$$\mathcal{L}_\beta^\downarrow \mathcal{L}_\beta^\uparrow = \mathcal{C}_{w_\beta^\downarrow,\phi^\downarrow}\mathcal{C}_{w_\beta^\uparrow,\phi^\uparrow} = \mathcal{C}_{w_\beta^\uparrow\circ\phi^\downarrow,\phi^\uparrow\circ\phi^\downarrow} = \mathcal{C}_{w_\beta^\uparrow\circ\phi^\downarrow,\mathrm{id}}\mathcal{C}_{1,\phi^\uparrow\circ\phi^\downarrow}, \tag{40}$$

and, since $w_\beta^\uparrow \circ \phi^\downarrow \in H^\infty(\Pi)$, the operator $\mathcal{C}_{w_\beta^\uparrow\circ\phi^\downarrow,\mathrm{id}}$ is bounded, so $\mathcal{L}_\beta^\downarrow \mathcal{L}_\beta^\uparrow$, being the product of a bounded and a compact operator, is itself compact. □

An immediate consequence of the previous proposition is the following estimate for the *essential spectral radius* of $\mathcal{L}_{\beta,h}$ (see, for example, [19, Chap. I.4]).

Proposition 7 *Let $\Re\beta \geq 0$ and $h \in \mathbb{C}$. The essential spectral radius of the operator $\mathcal{L}_{\beta,h} = e^{-\beta h}\mathcal{L}_\beta^\uparrow + e^{\beta h}\mathcal{L}_\beta^\downarrow$ acting on $H^\infty(\Pi)$ is bounded above by $e^{|\Re(\beta h)|}$.*

Proof Recall that the essential spectral radius of an operator $\mathcal{A}$ can be computed as follows (see, for example, [19, Chap. I, Theorem 4.10])

$$r_{ess}(\mathcal{A}) = \lim_{n\to\infty}\left(\inf_{\mathcal{K}\text{ compact}} \|\mathcal{A}^n - \mathcal{K}\|\right)^{1/n}. \tag{41}$$

Expanding the n-th power of $\mathcal{L}_{\beta,h}$, we find

$$\mathcal{L}_{\beta,h}^n = e^{-n\beta h}\mathcal{L}_\beta^{\uparrow\,n} + e^{n\beta h}\mathcal{L}_\beta^{\downarrow\,n} + \mathcal{K}_n \tag{42}$$

where $\mathcal{K}_n$ is a sum of $2^n - 2$ compact operators. In order to see that they are compact, note that each them is a product involving the compact operator $\mathcal{L}_\beta^\downarrow\mathcal{L}_\beta^\uparrow$ and bounded operators of the form $\mathcal{L}_\beta^{\downarrow\,k}$ and $\mathcal{L}_\beta^{\uparrow\,l}$. Thus

$$r_{ess}(\mathcal{L}_{\beta,h}) \leq \limsup_{n\to\infty} \|e^{-n\beta h}\mathcal{L}_\beta^{\uparrow\,n} + e^{n\beta h}\mathcal{L}_\beta^{\downarrow\,n}\|^{1/n} \tag{43}$$

and hence, by Proposition 4,

$$r_{ess}(\mathcal{L}_{\beta,h}) \leq \max\left\{|e^{-\beta h}|r(\mathcal{L}_\beta^\uparrow), |e^{\beta h}|r(\mathcal{L}_\beta^\downarrow)\right\} = e^{|\Re(\beta h)|}. \tag{44}$$

□

It turns out that the operator $\mathcal{L}_{\beta,h}$ has a number of interesting spectral properties if $\beta \geq 0$ and $h \in \mathbb{R}$, because, in this case, $\mathcal{L}_{\beta,h}$ is a *positive* operator (see below). Exploiting this additional structure requires some more terminology, which we now review (for more background see [20] or [21]).

Let $H^\infty_{\mathbb{R}}(\Pi) = \{f \in H^\infty(\Pi) : f(z) \in \mathbb{R} \text{ for } z > 0\}$. This is a real Banach space when equipped with the norm inherited from $H^\infty(\Pi)$ and its canonical complexification equals $H^\infty(\Pi)$ (see [22, Lemma 5.2]). In $H^\infty_{\mathbb{R}}(\Pi)$ consider the cone $K = \{f \in H^\infty_{\mathbb{R}}(\Pi) : f(z) \geq 0 \text{ for } z > 0\}$. Notice that K is closed and reproducing, that is, $K - K = H^\infty_{\mathbb{R}}(\Pi)$. For $f, g \in H^\infty_{\mathbb{R}}(\Pi)$, we write $f \leq g$ to mean that $g - f \in K$, and this defines a partial order on $H^\infty_{\mathbb{R}}(\Pi)$. An operator $\mathcal{L}$ on $H^\infty_{\mathbb{R}}(\Pi)$ is said to be *positive* (with respect to the partial order induced by K) if $f \geq 0$ implies $\mathcal{L}f \geq 0$, or equivalently, if $\mathcal{L}$ leaves K invariant.

Lemma 8 *Let $\beta \geq 0$ and $h \in \mathbb{R}$. Then the operators $\mathcal{L}^\downarrow_\beta$, $\mathcal{L}^\uparrow_\beta$, and $\mathcal{L}_{\beta,h}$ are positive on $H^\infty_{\mathbb{R}}(\Pi)$ with respect to K.*

Proof Observe that $w^\uparrow_\beta \in K$. Thus, $f \in K$ implies $\mathcal{L}^\uparrow_\beta f \in K$ since $w^\uparrow_\beta(z) f(\phi^\uparrow(z)) \geq 0$ for $z > 0$. Hence $\mathcal{L}^\uparrow_\beta$ is positive. By a similar argument, $\mathcal{L}^\downarrow_\beta$ is positive. Finally, $\mathcal{L}_{\beta,h}$ is positive since it is a sum of positive operators. □

A consequence of positivity is the following lower bound for the spectral radius of $\mathcal{L}_{\beta,h}$.

Proposition 9 *Let $\beta \geq 0$ and $h \in \mathbb{R}$. Then the spectral radius of the operator $\mathcal{L}_{\beta,h} = e^{-\beta h}\mathcal{L}^\uparrow_\beta + e^{\beta h}\mathcal{L}^\downarrow_\beta$ acting on $H^\infty(\Pi)$ is bounded below by $e^{|\beta h|}$.*

Proof We start with the case $h \geq 0$. Since

$$\mathcal{L}_{\beta,h} 1 = e^{-\beta h} w^\uparrow_\beta + e^{\beta h} 1 \geq e^{\beta h} 1, \tag{45}$$

the bound $r(\mathcal{L}_{\beta,h}) \geq e^{\beta h}$ follows by [21, Lemma 9.1], and the positivity of $\mathcal{L}_{\beta,h}$.

Let now $h \leq 0$. For $t > 0$ define $f_t(z) := (z+t)^{-2\beta}$. Note that $f_t \in K$ for any $t > 0$. We now claim that for any $t > 0$

$$\mathcal{L}_{\beta,h} f_t \geq \frac{e^{-\beta h}}{(1+t)^{2\beta}} f_t. \tag{46}$$

In order to see this note that for $z > 0$

$$\mathcal{L}_{\beta,h} f_t(z) \geq e^{-\beta h}\mathcal{L}^\uparrow_\beta f_t(z) = \frac{e^{-\beta h}}{(z + t(1+z))^{2\beta}} \geq \frac{e^{-\beta h}}{(1+t)^{2\beta}} \frac{1}{(1+z)^{2\beta}} \tag{47}$$

where the last inequality follows since $(z + t(1+z)) \leq (1+t)(1+z)$ for $t, z > 0$. Now, as before, (46) and the positivity of $\mathcal{L}_{\beta,h}$ imply

$$r(\mathcal{L}_{\beta,h}) \geq \frac{e^{-\beta h}}{(1+t)^{2\beta}} \tag{48}$$

by [21, Lemma 9.1], which in turn yields $r(\mathcal{L}_{\beta,h}) \geq e^{-\beta h}$ by letting $t \to 0$. □

We now summarise what we know about the spectral properties of $\mathcal{L}_{\beta,h}$.

Theorem 10 *Let $\beta \geq 0$ and $h \in \mathbb{R}$. For the operator $\mathcal{L}_{\beta,h}$ acting on $H^\infty(\Pi)$ we have the bounds*

$$r_{ess}(\mathcal{L}_{\beta,h}) \leq e^{|\beta h|} \leq r(\mathcal{L}_{\beta,h}). \tag{49}$$

The spectrum of $\mathcal{L}_{\beta,h}$ *in the annulus* $\{z \in \mathbb{C} : |z| > e^{|\beta h|}\}$ *coincides with* $\Omega_{\beta,h}$ *and consists of isolated eigenvalues of finite algebraic multiplicity. Moreover, if* $r(\mathcal{L}_{\beta,h}) > r_{ess}(\mathcal{L}_{\beta,h})$ *then* $r(\mathcal{L}_{\beta,h})$ *is an eigenvalue of* $\mathcal{L}_{\beta,h}$.

Proof This follows from Proposition 9, Proposition 3 and the definition of the essential spectral radius. The last assertion follows from Lemma 8 and [21, Exercise 8.2]. □

We end this section with a number of results on the operators $\mathcal{M}^{\downarrow}_{\beta,\tau_\downarrow}\mathcal{M}^{\uparrow}_{\beta,\tau_\uparrow}$ which will be needed for the perturbative argument below. A short calculation shows that

$$\mathcal{M}^{\downarrow}_{\beta,\tau_\downarrow}\mathcal{M}^{\uparrow}_{\beta,\tau_\uparrow} f = \sum_{m,n=1}^{\infty} \tau_\downarrow^m \tau_\uparrow^n w_\beta^{(m,n)} \cdot f \circ \phi^{(m,n)}, \tag{50}$$

with

$$w_\beta^{(m,n)}(z) = \frac{1}{(nz+mn+1)^{2\beta}} \quad \text{and} \quad \phi^{(m,n)}(z) = \frac{z+m}{nz+mn+1}. \tag{51}$$

Proposition 11 *Let*

$$D_1 = \left\{(\beta, \tau_\downarrow, \tau_\uparrow) \in \mathbb{C}^3 : \Re\beta > 0, |\tau_\downarrow| < 1, |\tau_\uparrow| < 1\right\}, \tag{52}$$

$$D_2 = \left\{(\beta, \tau_\downarrow, \tau_\uparrow) \in \mathbb{C}^3 : \Re\beta > \frac{1}{2}, |\tau_\downarrow| \le 1, |\tau_\uparrow| \le 1\right\}. \tag{53}$$

The function $(\beta, \tau_\downarrow, \tau_\uparrow) \mapsto \mathcal{M}^{\downarrow}_{\beta,\tau_\downarrow}\mathcal{M}^{\uparrow}_{\beta,\tau_\uparrow}$ *has the following properties*:

(i) *on* D_1 *it is holomorphic in the operator norm topology*;
(ii) *on* D_2 *it is continuous in the operator norm topology*;
(iii) *on* $D_1 \cup D_2$ *its values are compact operators*.

Proof First observe that for $\Re\beta \ge 0$ and $z \in \Pi$

$$\left|w_\beta^{(m,n)}(z)\right| \le \frac{e^{\pi|\Im\beta|}}{|nz+mn+1|^{2\Re\beta}}, \tag{54}$$

so

$$\left\|w_\beta^{(m,n)}\right\| \le \frac{e^{\pi|\Im\beta|}}{(mn)^{2\Re\beta}}, \tag{55}$$

hence

$$\left\|\mathcal{M}^{\downarrow}_{\beta,\tau_\downarrow}\mathcal{M}^{\uparrow}_{\beta,\tau_\uparrow}\right\| \le \sum_{m,n=1}^{\infty} |\tau_\downarrow|^m |\tau_\uparrow|^n \frac{e^{\pi|\Im\beta|}}{(mn)^{2\Re\beta}}, \tag{56}$$

which means that the series (50) converges in the operator norm topology for any $(\beta, \tau_\downarrow, \tau_\uparrow) \in D_1 \cup D_2$.

Assertions (i) and (ii) now follow by observing that $\beta \mapsto w_\beta^{(m,n)}$ is holomorphic (and thus continuous) for $\Re\beta > 0$ in the norm topology on $H^\infty(\Pi)$ for every $m, n \in \mathbb{N}$. For the proof of (iii) we note that for any fixed $\Re\beta > 0$, $|\tau_\downarrow|, |\tau_\uparrow| < 1$

$$\mathcal{M}^{\downarrow}_{\beta,\tau_\downarrow}\mathcal{M}^{\uparrow}_{\beta,\tau_\uparrow} = \tau_\downarrow\tau_\uparrow[1-\tau_\downarrow\mathcal{L}^{\downarrow}_\beta]^{-1}\mathcal{L}^{\downarrow}_\beta\mathcal{L}^{\uparrow}_\beta[1-\tau_\uparrow\mathcal{L}^{\uparrow}_\beta]^{-1}, \tag{57}$$

so $\mathcal{M}^{\downarrow}_{\beta,\tau_\downarrow}\mathcal{M}^{\uparrow}_{\beta,\tau_\uparrow}$ is compact by Proposition 6. The remaining assertion now follows from (ii) and the fact that the operator norm limit of compact operators is itself a compact operator. □

Proposition 12 *For $\beta > \frac{1}{2}$, $0 < \tau_\downarrow, \tau_\uparrow \le 1$, the operator $\mathcal{M}^{\downarrow}_{\beta,\tau_\downarrow}\mathcal{M}^{\uparrow}_{\beta,\tau_\uparrow}$ has a simple leading eigenvalue.*

Proof Fix $\beta > \frac{1}{2}$, $0 < \tau_\downarrow, \tau_\uparrow \le 1$. Observe that $\mathcal{M}^{\downarrow}_{\beta,\tau_\downarrow}\mathcal{M}^{\uparrow}_{\beta,\tau_\uparrow}$ is a transfer operator corresponding to a real analytic full branch expanding map on [0, 1] with strictly positive weights. The proof of [22, Proposition 4.9] now shows that $\mathcal{M}^{\downarrow}_{\beta,\tau_\downarrow}\mathcal{M}^{\uparrow}_{\beta,\tau_\uparrow}$ is 1-positive with respect to K. Since $\mathcal{M}^{\downarrow}_{\beta,\tau_\downarrow}\mathcal{M}^{\uparrow}_{\beta,\tau_\uparrow}$ is compact by Proposition 11 the assertion now follows from [20, Theorems 2.5, 2.10 and 2.13] and the fact that the canonical complexification of $H^\infty_{\mathbb{R}}(\Pi)$ is $H^\infty(\Pi)$. □

5 Dynamical Systems

In this section we provide explicit representations of the operators defined above and show the connection to the Gauss map.

The operators $\mathcal{M}^{\uparrow}_{\beta,\tau}$, $\mathcal{M}^{\downarrow}_{\beta,\tau}$, and $\mathcal{M}_{\beta,\tau}$ have explicit power series expansions in τ, given by

$$\mathcal{M}^{\uparrow}_{\beta,\tau} f(x) = \sum_{n=1}^{\infty} \frac{\tau^n}{(1+nx)^{2\beta}} f\left(\frac{x}{1+nx}\right), \qquad \mathcal{M}^{\downarrow}_{\beta,\tau} f(x) = \sum_{n=1}^{\infty} \tau^n f(x+n), \tag{58}$$

and

$$\mathcal{M}_{\beta,\tau} f(x) = \sum_{n=1}^{\infty} \frac{\tau^n}{(n+x)^{2\beta}} f\left(\frac{1}{n+x}\right). \tag{59}$$

In order to find an interpretation of these operators as weighted transfer operators associated to interval maps, note that the operators $\mathcal{M}^{\uparrow}_{\beta,\tau}$ and $\mathcal{M}^{\downarrow}_{\beta,\tau}$ are given by the collection of the left-most and right-most branches, respectively, of the iterated transformations T^n for $n \in \mathbb{N}$, where T is given in (15). This is indicated in Fig. 2.

If one restricts the maps as indicated by the solid lines in Fig. 2, one sees that another interval map $\hat{T}$ is formed on $\mathbb{R}^+$. Clearly the map $\hat{T}$ exchanges the intervals $(0, 1)$ and $(1, \infty)$.

Recall that the operator $\mathcal{M}_{\beta,\tau}$ was defined by composition with $\mathcal{S}_\beta$ as

$$\mathcal{M}_{\beta,\tau} = \mathcal{M}^{\downarrow}_{\beta,\tau}\mathcal{S}_\beta = \mathcal{S}_\beta\mathcal{M}^{\uparrow}_{\beta,\tau}. \tag{60}$$

The transformation underlying the operator $\mathcal{M}_{\beta,\tau}$ is obtained from Fig. 2 by composing $\hat{T}$ with $S(x) = 1/x$. It is easy to see that $\hat{T}S = S\hat{T}$. The graph of $\hat{T}S$ is shown in Fig. 3. The associated dynamical system is split into two independent subsystems on the intervals $(0, 1)$ and $(1, \infty)$, respectively.

Restricting $\mathcal{M}_{\beta,1}$ to act on functions on the unit interval [0, 1] gives precisely the Ruelle-Perron-Frobenius transfer operator of the Gauss map $x \mapsto 1/x \mod 1$, which has branches

$$\hat{T}_n(x) = \frac{1}{x} - n. \tag{61}$$

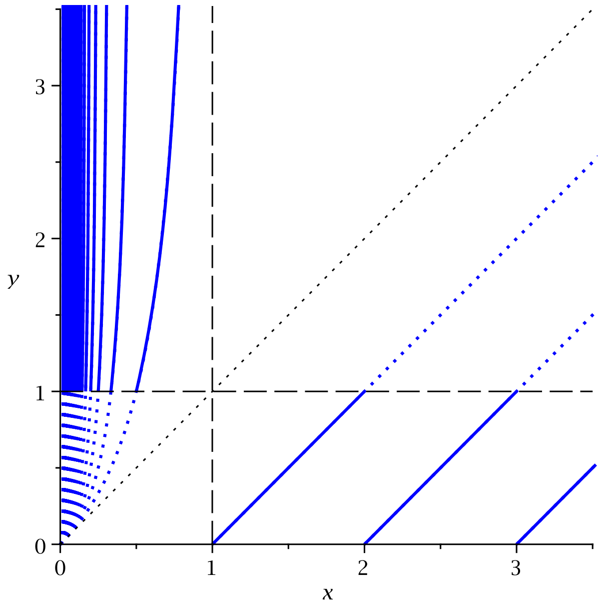


Fig. 2 The graph of the left-most and right-most branches of the iterated transformations T^n for $n \in \mathbb{N}$. Restricting the iterates as indicated by *the solid lines*, one obtains the interval map $y = \hat{T}(x)$

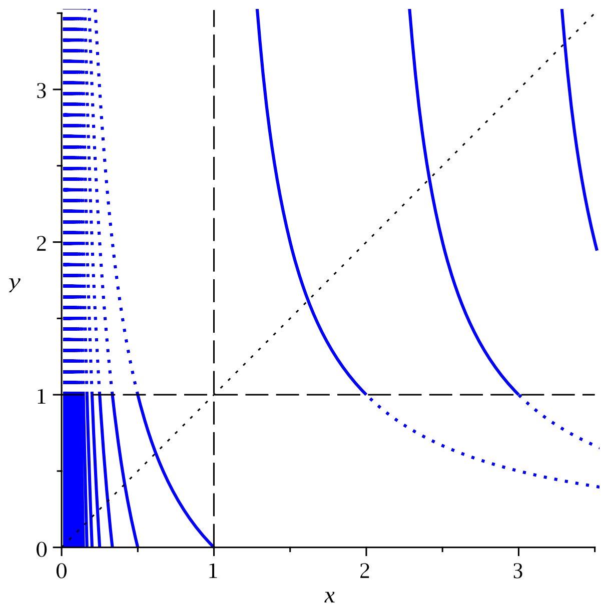


Fig. 3 The graph of $y = \hat{T}S(x)$ (*solid lines*), along with the analytic extension of its branches (*dotted lines*)

Accordingly, $\mathcal{M}_{\beta,\tau}$ may be regarded as a generalised transfer operator of the Gauss map, with the branches weighted differently.

This interpretation will be a key ingredient in the perturbative expansion in the next section.

6 Perturbation Theory

In this section we come to the final point of our analysis. We employ Proposition 2, which connects eigenvalues of $\mathcal{M}_{\beta,ze^{\beta h}}\mathcal{M}_{\beta,ze^{-\beta h}}$ to eigenvalues of $\mathcal{L}_{\beta,h}$ to make a perturbation expansion around the critical point $(\beta,h)=(1,0)$. More explicitly, for β in a left neighbourhood of 1, we will choose z and h so that $\mathcal{M}_{\beta,ze^{\beta h}}\mathcal{M}_{\beta,ze^{-\beta h}}$ has eigenvalue 1. This results in an implicit equation for the inverse eigenvalue $z(\beta,h)$ which by (11) leads to the asymptotic form of the free energy $f(\beta,h)$.

The central object in the perturbative calculation is the operator product

$$\mathcal{P}_{\beta,ze^{\beta h},ze^{-\beta h}}=\mathcal{M}_{\beta,ze^{\beta h}}\mathcal{M}_{\beta,ze^{-\beta h}}. \tag{62}$$

By Propositions 11 and 12, we know that near the critical point, i.e. $z=1$, $\beta=1$, and $h=0$, this is a holomorphic compact operator-valued function that extends continuously to this point. Furthermore, at the critical point, $\mathcal{P}_{1,1,1}$ has a simple leading eigenvalue.

To keep notational overload to a minimum, we write

$$\vec{\tau}=(\tau_\downarrow,\tau_\uparrow)=(ze^{\beta h},ze^{-\beta h}) \tag{63}$$

and put $\mathcal{P}_{\beta,\vec{\tau}}=\mathcal{P}_{\beta,ze^{\beta h},ze^{-\beta h}}=\mathcal{M}_{\beta,ze^{\beta h}}\mathcal{M}_{\beta,ze^{-\beta h}}$. We further omit any of the variables $(\beta,\tau_\downarrow,\tau_\uparrow)$ when they take on their respective value at the critical point, i.e. $z=1$, $\beta=1$, or $h=0$. For example, we write $\mathcal{P}=\mathcal{P}_{1,1,1}$, $\mathcal{P}_{\vec{\tau}}=\mathcal{P}_{1,ze^{h},ze^{-h}}$ and so on. We use similar conventions for other quantities.

Now we find formally that

$$\mathcal{L}h=h \quad \text{with } h(x)=\frac{1}{x}, \tag{64}$$

however the function h is not bounded, and therefore not an eigenfunction of the operator in the space $H^\infty(\Pi)$.

On the other hand, the corresponding equation for $\mathcal{P}=\mathcal{P}_{1,1,1}$ is

$$\mathcal{P}g=g \quad \text{with } g(x)=\frac{1}{\log(2)}\frac{1}{1+x}, \tag{65}$$

and we can check easily that indeed formally $h=\log(2)(1+\mathcal{M}^{\uparrow}_{1,1})g$ as expected. The function g lies in $H^\infty(\Pi)$, and is therefore an eigenfunction of $\mathcal{P}$.

As indicated in the previous section, $\mathcal{P}$ is the Perron-Frobenius operator of the second iterate of the Gauss map on the unit interval. It follows that the left eigenfunction of $\mathcal{P}$ is $\mu=\mu_L$, the Lebesgue measure on [0, 1].

Integration of $1/(1+x)$ with respect to the Lebesgue measure on [0, 1] gives log(2), which motivates the normalisation of g.

Given the eigenvalue equations

$$\mathcal{P}_{\beta,\vec{\tau}}g_{\beta,\vec{\tau}}=\lambda_{\beta,\vec{\tau}}g_{\beta,\vec{\tau}}, \qquad \mu_{\beta,\vec{\tau}}\mathcal{P}_{\beta,\vec{\tau}}=\lambda_{\beta,\vec{\tau}}\mu_{\beta,\vec{\tau}}, \tag{66}$$

we shall now solve the equation

$$\lambda_{\beta,\vec{\tau}}=1 \tag{67}$$

perturbatively around $\beta=1$ and $\vec{\tau}=(1,1)$, proceeding as in [23].

By Proposition 11, the compact operator $\mathcal{P}_{\beta,\vec{\tau}}$ is an analytic function of β for $\Re\beta > 0$ in the operator norm topology. Thus, we can expand $\mathcal{P}_{\beta,\vec{\tau}}$ around $\beta = 1$ as

$$\mathcal{P}_{\beta,\vec{\tau}} = \mathcal{P}_{\vec{\tau}} + \sum_{n=1}^{\infty}(1-\beta)^n \mathcal{P}_{\vec{\tau}}^{(n)}. \tag{68}$$

Moreover, since by Proposition 12 the leading eigenvalue $\lambda_{\beta,\vec{\tau}}$ is simple, therefore it is analytic in β and continuous in $\vec{\tau}$. By the same argument, $g_{\beta,\vec{\tau}}$ ($\mu_{\beta,\vec{\tau}}$) is holomorphic in β and continuous in $\vec{\tau}$ with respect to the norm topology on $H^\infty(\Pi)$ (the strong dual topology on the dual of $H^\infty(\Pi)$). We thus have expansions analogous to (68) for the eigenvalues $\lambda_{\beta,\vec{\tau}}$ and the left and right eigenfunctions $g_{\beta,\vec{\tau}}$ and $\mu_{\beta,\vec{\tau}}$, respectively.

We choose the normalisation $\mu_{\beta,\vec{\tau}} g_{\beta,\vec{\tau}} = 1$. Expanding

$$\mu_{\vec{\tau}} \mathcal{P}_{\beta,\vec{\tau}} g_{\beta,\vec{\tau}} = \lambda_{\beta,\vec{\tau}} \mu_{\vec{\tau}} g_{\beta,\vec{\tau}} \tag{69}$$

and

$$\mu_{\beta,\vec{\tau}} \mathcal{P}_{\beta,\vec{\tau}} g_{\vec{\tau}} = \lambda_{\beta,\vec{\tau}} \mu_{\beta,\vec{\tau}} g_{\vec{\tau}} \tag{70}$$

to lowest orders in $(1-\beta)$ and comparing coefficients, we find for the first-order change of the eigenvalue

$$\lambda_{\vec{\tau}}^{(1)} = \mu_{\vec{\tau}} \mathcal{P}_{\vec{\tau}}^{(1)} g_{\vec{\tau}}, \tag{71}$$

which is a standard result of first-order perturbation theory [24]. We therefore have

$$\lambda_{\beta,\vec{\tau}} = \lambda_{\vec{\tau}} + (1-\beta)\mu_{\vec{\tau}} \mathcal{P}_{\vec{\tau}}^{(1)} g_{\vec{\tau}} + O((1-\beta)^2) \tag{72}$$

$$= \lambda_{\vec{\tau}} + (1-\beta)\mu \mathcal{P}^{(1)} g[1 + o(1-\tau_\uparrow) + o(1-\tau_\downarrow)] + O((1-\beta)^2), \tag{73}$$

where for the final estimate we have used continuity in $\vec{\tau}$.

From $\mathcal{P}_\beta = \mathcal{M}_\beta^2$, where $\mathcal{M}_\beta = \mathcal{M}_{\beta,1}$ is the transfer operator for the Gauss map, it follows that

$$\mu \mathcal{P}^{(1)} g = \mu \mathcal{M}^{(1)} \mathcal{M} g + \mu \mathcal{M} \mathcal{M}^{(1)} g = 2\mu \mathcal{M}^{(1)} g, \tag{74}$$

where we have expanded $\mathcal{M}_\beta = \mathcal{M} + (1-\beta)\mathcal{M}^{(1)} + O((1-\beta)^2)$.

By a standard result [15], this can be expressed in terms of the Lyapunov exponent of the associated interval map. Here, one obtains (see e.g. [25]) the Lyapunov exponent λ_G of the Gauss map,

$$\lambda_G = \mu \mathcal{M}^{(1)} g = -\frac{\partial}{\partial\beta}\bigg|_{\beta=1} \mu \mathcal{M}_\beta g = \frac{\pi^2}{6\log(2)}. \tag{75}$$

Therefore (73) gives

$$\lambda_{\beta,\vec{\tau}} = \lambda_{\vec{\tau}} + 2\lambda_G(1-\beta)[1 + o(1-\tau_\uparrow) + o(1-\tau_\downarrow)] + O((1-\beta)^2). \tag{76}$$

Next, we consider the $\vec{\tau}$ dependence of $\lambda_{\vec{\tau}} = \lambda_{1,\vec{\tau}}$. Using $\lambda_{\vec{\tau}} \mu g_{\vec{\tau}} = \mu \mathcal{P}_{\vec{\tau}} g_{\vec{\tau}}$, we rewrite

$$\mu \mathcal{P}_{\vec{\tau}} g_{\vec{\tau}} - \mu \mathcal{P} g_{\vec{\tau}} = (\lambda_{\vec{\tau}} - 1)\mu g_{\vec{\tau}}. \tag{77}$$

Hence

$$\lambda_{\vec{\tau}} = 1 + \frac{\mu(\mathcal{P}_{\vec{\tau}} - \mathcal{P})g_{\vec{\tau}}}{\mu(g_{\vec{\tau}})} = 1 + \frac{\mu(\mathcal{P}_{\vec{\tau}} - \mathcal{P})g + \mu(\mathcal{P}_{\vec{\tau}} - \mathcal{P})(g_{\vec{\tau}} - g)}{1 + \mu(g_{\vec{\tau}} - g)}, \tag{78}$$

where we have used the normalisation condition $\mu g = 1$. Continuity in $\vec{\tau}$ implies that

$$\lambda_{\vec{\tau}} = 1 + \mu(\mathcal{P}_{\vec{\tau}} - \mathcal{P})g[1 + o(1-\tau_\uparrow) + o(1-\tau_\downarrow)]. \tag{79}$$

Combining (67), (76), and (79), this implies

$$-\mu(\mathcal{P}_{\vec{\tau}} - \mathcal{P})g \sim 2\lambda_G(1-\beta) \tag{80}$$

as $\beta \to 1$ (and, hence, both $\tau_\uparrow \to 1^-$ and $\tau_\downarrow \to 1^-$).

The final step lies in the estimate of $\mathcal{P}_{\vec{\tau}} - \mathcal{P}$. We write

$$\mathcal{P}_{\vec{\tau}} - \mathcal{P} = \mathcal{M}_{\tau_\downarrow}\mathcal{M}_{\tau_\uparrow} - \mathcal{M}^2 = (\mathcal{M}_{\tau_\downarrow} - \mathcal{M})\mathcal{M} + \mathcal{M}(\mathcal{M}_{\tau_\uparrow} - \mathcal{M}) + (\mathcal{M}_{\tau_\downarrow} - \mathcal{M})(\mathcal{M}_{\tau_\uparrow} - \mathcal{M}). \tag{81}$$

Equation (59) implies that

$$\|\mathcal{M}_\tau - \mathcal{M}\| \le \eta(\tau). \tag{82}$$

Here $\eta(\tau) = \sum_{n=1}^\infty \frac{1-\tau^n}{n^2} = \mathrm{Li}_2(1) - \mathrm{Li}_2(\tau)$, where Li_2 denotes the dilogarithm. It follows immediately that $\|(\mathcal{M}_{\tau_\downarrow} - \mathcal{M})(\mathcal{M}_{\tau_\uparrow} - \mathcal{M})\| \le \eta(\tau_\downarrow)\eta(\tau_\uparrow)$. Hence,

$$\mathcal{P}_{\vec{\tau}} - \mathcal{P} = (\mathcal{M}_{\tau_\downarrow} - \mathcal{M})\mathcal{M} + \mathcal{M}(\mathcal{M}_{\tau_\uparrow} - \mathcal{M}) + O(\eta(\tau_\downarrow)\eta(\tau_\uparrow)) \tag{83}$$

in operator norm. Hence, by (80),

$$-\mu(\mathcal{M}_{\tau_\downarrow} - \mathcal{M})g + \mu(\mathcal{M}_{\tau_\uparrow} - \mathcal{M})g \sim 2\lambda_G(1-\beta). \tag{84}$$

An explicit calculation then gives the exact expression

$$\mu(\mathcal{M}_\tau - \mathcal{M})g = -\frac{(1-\tau)^2}{\tau^2\log(2)}\sum_{n=1}^\infty \tau^n \log n. \tag{85}$$

The asymptotic form follows on writing

$$\log n = \sum_{k=1}^n \frac{1}{k} - \gamma - \frac{1}{2n} + O(n^{-2}), \tag{86}$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant. Inserting this in (85) gives immediately

$$\sum_{n=1}^\infty \tau^n \log n = \frac{1}{1-\tau}\log\frac{1}{1-\tau} - \gamma\frac{1}{1-\tau} - \frac{1}{2}\log\frac{1}{1-\tau} + O(1), \tag{87}$$

uniformly for $|\tau| < 1$.

We are now in a position to obtain the asymptotic expansion of the free energy. Inserting the leading order asymptotics of $\mu(\mathcal{M}_\tau - \mathcal{M})g$ into (84), we arrive at

$$[-(1-\tau_\downarrow)\log(1-\tau_\downarrow) - (1-\tau_\uparrow)\log(1-\tau_\uparrow)] \sim 2\log(2)\lambda_G(1-\beta). \tag{88}$$

Substituting $\tau_\downarrow = e^{\beta(f+h)}$ and $\tau_\uparrow = e^{\beta(f-h)}$ and expanding for small f and h gives to leading order

$$2\log(2)\lambda_G(1-\beta) \sim (f+h)\log(-(f+h)) + (f-h)\log(-(f-h)). \tag{89}$$

Setting $\beta_c = 1$ and $C = \log(2)\lambda_G = \pi^2/6$, we see that this is, aside from constants, the same as (40) in [7], which was found using a cluster approximation. The analysis therein then immediately gives (cf. (46) in [7])

$$f \sim \frac{t}{\log t} - \frac{1}{2}\frac{h^2}{t} \quad \text{for } h^2 \ll t \ll 1, \tag{90}$$

where the rescaled temperature variable t is given by $t = 2\log(2)\lambda_G(1-\beta)$. Therefore the temperature deviation from the critical point is scaled by the Lyapunov exponent of the Gauss map, $\lambda_G = \frac{\zeta(2)}{\log(2)} = \frac{\pi^2}{6\log(2)}$. Note that, in addition, (90) implies that λ_G determines the amplitude of both the specific heat C and susceptibility χ singularities, with C proportional to $1/\lambda_G$ and χ proportional to λ_G.

Also, just as in [7], the asymptotic shape of the phase boundary is given by letting $-f = |h| = h_c$ in (89). We obtain for the dependence of the critical field strength h_c to leading order

$$h_c \sim \frac{t}{\log t}. \tag{91}$$

7 Discussion

It is of interest to discuss the relation of the present treatment to the cluster approximation presented in [7]. In that approximation the central quantity is the cluster generating function $\Lambda(\beta,\tau)$. Here we do not linearise the dynamical map, with the result that the function $\Lambda(\beta,\tau)$ is replaced by the operator $\mathcal{M}_{\beta,\tau}$. Conversely, the effect of linearising the map on the operator is that it becomes a multiplication operator when acting on constant functions.

Physically, this corresponds to replacing a complicated system with interactions of all types (see [6] and references therein) by non-interacting clusters. The significance of our work lies in the fact that the behaviour of both models near the critical point is identical, thus justifying the cluster approximation. Note that the resulting non-interacting cluster model is similar to the ones discussed in [3–5].

As mentioned, the renormalisation group result for $f(\beta,h)$ found in [13] does not quite agree with (90). Specifically, the second term has the form $\frac{h^2\log t}{t}$, which as $t \to 0$, is larger than the corresponding term in (90). As discussed in [13], there does not seem to be any consistent way to remove this term in the renormalisation group framework. However, this is perhaps not so surprising, since the Farey model is known to have long-range interactions (see [6] and references therein), which renders results from a renormalisation group treatment questionable.

Finally, although our results are exact, we indicate which points of our treatment are not quite rigorous. The results given in Sect. 3 are rigorous, but the problem of pointwise evaluation is not completely settled, and the spectrum of $\mathcal{P}_{\beta,\bar{\tau}}$ has not been fully characterised. In particular, the possibility of another leading eigenvalue has not be ruled out. In addition, our particular choice of a function space might seem unusual in that it does not respect the "spin flip" symmetry of the model. However, it does not seem possible to find a function space respecting this symmetry for which the perturbation calculations employed are tractable.

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